DISPERSION OF A SUBSTANCE IN NARROW CHANNELS IN A LUBRICANT LAYER

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An approach commonly taken to studying problems involving heat and mass transfer in the flow of a fluid in channels of different shapes is reduction of the basic equations of convective diffusion (heat conduction) to a dispersion equation which has one less independent variable than the initial equations and, as a rule, has constant coefficients. This approach was first successfully used in the study by J. Taylor [1]. Dispersion theory is now quite advanced and has been the subject of an anormous number of publications (see the survey in [2]).

Most investigators have studied the dispersion of substances for flows with simple velocity profiles that are usually unidimensional. Such problems clearly do not embrace the entire range of hydrodynamic situations encountered under natural conditions and in industrial equipment. For example, when a study is being made of dispersion in curvilinear channels, researchers usually restrict themselves to problems in toroidal channels [3-5]. In this case, it is possible to use the corresponding formulas for the velocity components from the known solution of the hydrodynamic problem. Also of interest from a practical standpoint are problems involving the propagation of heat and mass in an approximation of dispersion theory for the channels typically encountered in studies of flows of lubricant layers in bearings. A considerable amount of progress has been made with regard to the hydrodynamic part of the problem in lubrication theory, and we will proceed on this basis. For the sake of definiteness, below we will concern outselves only with diffusion problems – although it should be clear that the results can easily be applied to other heat transfer problems.

The equations of the hydrodynamic theory of lubrication are the limiting form of the Navier–Stokes equations in the case when the ratio of the characteristic dimensions of the region in different directions is small and (usually) the Reynolds numbers are low [6, 7]. We will use the methods of pertubation theory [8, 9] to analyze the equation of convective diffusion, here employing the same small parameter as in lubrication theory. In each case considered below, we assume that the velocity field of the fluid is known. It is further assumed that the field is not necessarily formed by the same mechanisms as in problems concerning the hydrodynamics of a lubricant layer, i.e., the mass transfer equations obtained below have a more general value.

A characteristic of the mass transfer regions we will examine is that, in the general case, the boundaries of the regions do not coincide with the coordinate lines of any certain coordinate system. However, the deviations of the given region from a "rectangular" shape in the corresponding coordinates are insignificant, as is typical for problems of the hydrodynamic theory of lubrication. It will be expedient to conduct our study in the three most widely used coordinate systems: Cartesian, cylindrical, and spherical.

1. Dispersion of a Substance in a Layer above a Flat Surface. It is natural to analyze the given case in a Cartesian coordinate system. The flow region is shown in Fig. 1. We will write the main equation of convective diffusion in dimensionless form

$$\varepsilon^2\left(\frac{\partial c}{\partial t}+u_x^*\frac{\partial c}{\partial x}\right)+\varepsilon\mathrm{Pe}\left(u_x\frac{\partial c}{\partial x}+u_y\frac{\partial c}{\partial y}\right)=\frac{\partial^2 c}{\partial y^2}+\varepsilon^2\frac{\partial^2 c}{\partial x^2}, \qquad (1.1)$$

where c is the concentration of the substance and ε is a small parameter (Fig. 1). Regarding the ratio of the characteristic scales of length in the Y- and X directions and the scales of the velocity components U_y and U_x, they are of the same order ε as in lubrication theory

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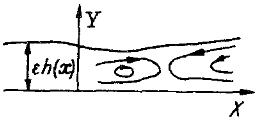

Fig. 1

[6]. Then,

$$x = \frac{X}{l},\ y = \frac{Y}{l\varepsilon},\ t = \frac{D\tau}{l^2},\ \mathrm{Pe} = \frac{Ul\varepsilon}{D},\ u_x^* = \frac{\langle U_x\rangle l}{D}.$$

Here, X and Y are dimensional coordinates; ℓ is the characteristic length in the X direction; D is the diffusion coefficient; τ is time; U is the characteristic velocity (scale) along the X axis; Pe is the Peclet number, this being assumed an independent parameter having the order of unity with respect to ε; u_x^* is the dimensionless mean velocity; $\langle U_x\rangle$ is mean flow velocity; and $u_x = (U_x - \langle U_x\rangle)/U$.

Convective transport of a substance associated with the mean flow velocity is usually excluded by changing over to a longitudinal coordinate which moves with mean velocity. In our case, mean velocity is a function of x and convective transport could have been excluded by means of the "characteristic" $\rho = \int\limits_0^x d\xi/u_x^*(\xi) - t$. However, in light of the fact that the corresponding term in the convective diffusion equation has the order ε^2, formal calculations produce the same result with the reverse transition to a "stationary" coordinate system. Thus, to be consistent, we should use the continuity equation in the form

$$\varepsilon\, du_x^*/dx + \mathrm{Pe}\,(\partial u_x/\partial x + \partial u_y/\partial y) = 0.$$

This has no effect on the result in the zeroth approximation with respect to ε, i.e., it has no effect on the form of the dispersion equation, since the term $\varepsilon du_x^*/dx$ can be ignored. It should also be noted that the choice of the scale for time ℓ^2/D and the order of Pe are consistent with Taylor's method of successive approximation [1].

Equation (1.1) is supplemented by the initial condition

$$c\,|_{t=0} = c_*(x, y) \tag{1.2}$$

and the boundary conditions

$$\left.\frac{\partial c}{\partial y}\right|_{y=0} = 0,\ \left.\frac{\partial c}{\partial y}\right|_{y=h(x)} = \varepsilon^2 h'(x)\left.\frac{\partial c}{\partial x}\right|_{y=h(x)}, \tag{1.3}$$

expressing the absence of diffusional transport of the substance to the flow region with the condition of vanishing of fluid velocity at the bounding surfaces (there is no convective transport at the boundaries). The conditions with respect to the variable x are trivial for our analysis and will thus not be specified.

As was shown in [10-12], in problems of the given type it is sufficient to limit oneself to three approximations with respect to ε in the expansion

$$c = c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \ldots \tag{1.4}$$

in order to derive the dispersion equation of the zeroth approximation. Insertion of (1.4) into (1.1) and (1.3) and grouping of terms of the same order with respect to ε gives us equations for the functions c_0, c_1, and c_2. Meanwhile, the first two equations lead to the conclusion that c_0 is a function only of x and t (and can henceforth be denoted as c_0 = G(x, t)), while the relation for c_1 can be represented (after a single integration of the corresponding equation over y) in the form

$$\frac{\partial c_1}{\partial y} = \mathrm{Pe}\,\Phi_x(x, y)\frac{\partial G}{\partial x},\ \Phi_x(x, y) = \int\limits_0^y u_x(x, y)\, dy. \tag{1.5}$$

Here, by separating the mean velocity from the overall profile, we obtain $\Phi_x(x, h) = 0$ to within terms of the order of ε. This follows from the continuity equation and results in agreement with first-order boundary conditions (1.3). The equation for the function c_2

$$\frac{\partial G}{\partial t}+u_x^*\frac{\partial G}{\partial x}+\mathrm{Pe}\left(u_x\frac{\partial c_1}{\partial x}+u_y\frac{\partial c_1}{\partial y}\right)=\frac{\partial^2 c_2}{\partial y^2}+\frac{\partial^2 G}{\partial x^2}$$

is integrated over y within the limits $[0, h(x)]$. We also use the corresponding condition (1.3)[$(\partial c_2/\partial y = h'(x)\partial G/\partial x$ at $y = h(x))$], the continuity equation, and the relation for c_1. As a result, we find that

$$\frac{\partial G}{\partial t}+u_x^*\frac{\partial G}{\partial x}=\frac{1}{h(x)}\frac{\partial}{\partial x}\left\{h(x)[\mathrm{Pe}^2D_x(x)+1]\frac{\partial G}{\partial x}\right\}, \tag{1.6}$$

where

$$D_x(x)=\frac{1}{h(x)}\int_0^{h(x)}\Phi_x^2(x, y)\,dy \tag{1.7}$$

is the dimensionless dispersion factor (the convective part). The "molecular" part of the factor is influenced by the geometry of the region (the function $h(x)$), which was made variable. It should be noted that, in the zeroth approximation with respect to ε, the function G is the concentration of the substance averaged over the cross section.

The initial condition for Eq. (1.6) can be obtained after constructing an internal [8, 9] expansion and combining the solutions of the internal and external problems. Following [10-12], we obtain

$$G|_{t=0}=G_0(x)=\frac{1}{h(x)}\int_0^{h(x)}c_*(x, y)\,dy, \tag{1.8}$$

i.e., in the zeroth approximation for ε initial condition (1.2) was subjected to simple averaging.

2. Dispersion of a Substance in a Cylindrical Gap. In contrast to part 1, we will make use of a cylindrical coordinate system for the present case. One of the boundaries of the region is the coordinate line $r = R$, while the second is the line $r = R[1 + \varepsilon h(\varphi)]$ (Fig. 2). Introducing the transverse coordinate $\zeta = (r - R)/\varepsilon R$, $\zeta \in [0, h(\varphi)]$, we write the equation of convective diffusion in the dimensionless form

$$\varepsilon^2\left(\frac{\partial c}{\partial t}+\frac{u_\varphi^*}{1+\varepsilon\zeta}\frac{\partial c}{\partial\varphi}\right)+\varepsilon\,\mathrm{Pe}\left(\frac{u_\varphi}{1+\varepsilon\zeta}\frac{\partial c}{\partial\varphi}+u_r\frac{\partial c}{\partial\zeta}\right)= \tag{2.1}$$

$$=\frac{\partial^2 c}{\partial\zeta^2}+\frac{\varepsilon}{1+\varepsilon\zeta}\frac{\partial c}{\partial\zeta}+\frac{\varepsilon^2}{(1+\varepsilon\zeta)^2}\frac{\partial^2 c}{\partial\varphi^2}$$

$$(\mathrm{Pe}=\varepsilon UR/D,\ t=D\tau/R^2,\ u_\varphi^*=\langle U_\varphi\rangle R/D).$$

As in part 1, Eq. (2.1) makes it relatively easy to obtain a dispersion equation of the following form for the function of the zeroth approximation $c_0 = g(\varphi, t)$ of an expansion of type (1.4)

$$\frac{\partial G}{\partial t}+u_\varphi^*\frac{\partial G}{\partial\varphi}=\frac{1}{h(\varphi)}\frac{\partial}{\partial\varphi}\left\{h(\varphi)[\mathrm{Pe}^2D_\varphi(\varphi)+1]\frac{\partial G}{\partial\varphi}\right\}, \tag{2.2}$$

where the convective part of the dispersion factor D_φ is determined by Eq. (1.7) with replacement of the variable and subscript x by φ. In essence, we have obtained an equation which coincides with (1.6). Similarly, (1.8) is analogous to the initial condition for (2.2).

Equations (1.6) and (2.2) can be reduced to self-adjoint form after replacement of the sought function. Then, after separation of the variables and formulation of the corresponding Sturm–Liouville problems, we can construct the solutions of sufficiently general problems by the Fourier method or its generalization for inhomogeneous problems. However, in the general case we will not encounter the normally-studied ordinary differential equations of the Sturm–Liouville problem and the corresponding special functions. For (2.2), one typically encounters problems with solutions that are periodic with respect to φ. Furthermore,

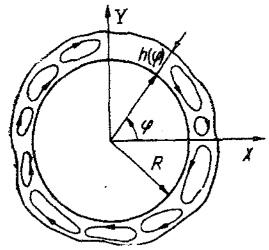

Fig. 2

since the coefficients in (2.2) are periodic with respect to φ, we obtain relatively little-studied problems for equations with periodic coefficients (see [13], for example).

Since $u_\varphi{}^*h$ is a constant equal to the rate of fluid flow through the layer, steady-state equation (2.2) – as (1.6) – is integrated directly and then reduced to a simple formula of integration. It is easily shown that the steady-state solution of (2.2), satisfying the conditions of periodicity, can only be constant. The value of this constant is easily found from the momentum conservation law for the system

$$\frac{d}{dt}\int_0^{2\pi} G(\varphi, t)\, h(\varphi)\, d\varphi = 0 \Rightarrow \int_0^{2\pi} G(\varphi, t)\, h(\varphi)\, d\varphi = \int_0^{2\pi} G_0(\varphi)\, h(\varphi)\, d\varphi, \tag{2.3}$$

obtained from (2.2) after multiplication by h, integration over φ for the period, and use of the periodicity conditions. Then by passing to the limit $t \to \infty$ in the second equation of (2.3) we obtain the steady-state solution

$$G_\infty = \int_0^{2\pi} G_0(\varphi)\, h(\varphi)\, d\varphi \Big/ \int_0^{2\pi} h(\varphi)\, d\varphi, \tag{2.4}$$

which is dependent on the initial function $G_0(\varphi)$.

In the special case h = const and when u_x in (1.1) and u_φ in (2.1) can be considered functions only of the transverse flow, the coordinates and the values of the dispersion factors will be independent of x (or φ), i.e., dispersion equations (1.6) and (2.2) will contain constant coefficients. By choosing dimensionless coordinates and parameters, we can represent these equations in the form

$$\partial G/\partial t + n\partial G/\partial\varphi = \partial^2 G/\partial\varphi^2. \tag{2.5}$$

Equation (2.5) is frequently encountered in the study of different heat- and mass transfer processes, and its solutions for various auxiliary conditions (the Dankwerst condition, etc.) have been well-studied (see [14]). However, problems involving mass transfer in a cylindrical gap are typified by nontraditional conditions corresponding to periodicity of the solution with respect to the cyclic coordinate φ:

$$G(0, t) = G(2\pi, t),\ \partial G/\partial\varphi|_{\varphi=0} = \partial G/\partial\varphi|_{\varphi=2\pi}. \tag{2.6}$$

Thus, we will obtain its solution with the usual initial condition (18) (with the replacement of by by φ). It is easily proven that problem (1.8), (2.5-2.6) satisfies the expression

$$G(\varphi, t) = \frac{1}{2\pi}\int_0^{2\pi} G_0(\varphi)\, d\varphi + \sum_{k=1}^{\infty} \{M_k \sin[k(\varphi - nt)] + N_k \cos[k(\varphi - nt)]\} \exp(-k^2 t), \tag{2.7}$$

where M_k and N_k are the Fourier coefficients of the function $G_0(\varphi)$:

$$M_k = \frac{1}{\pi}\int_0^{2\pi} G_0(\varphi)\sin(k\varphi)\, d\varphi, \quad N_k = \frac{1}{\pi}\int_0^{2\pi} G_0(\varphi)\cos(k\varphi)\, d\varphi.$$

As might be expected, at $t \to \infty$, the solution of (2.7) approaches the mean value of the initial concentration $G_0(\varphi)$, i.e., it agrees with (2.4) at $h = \text{const}$.

3. Dispersion of a Substance in a Spherical Gap. In this case, it is most convenient to conduct the study in spherical coordinates, since the internal boundary of the region is the coordinate line $r = R$. For the external boundary, we have $r = R[1 + \varepsilon h(\varphi, \theta)]$ (Fig. 3). As in part 2, due to the condition $\varepsilon \ll 1$, it is also expedient to introduce the coordinate $\zeta = (r - R)/\varepsilon R$, $\zeta \in [0, h(\varphi, \theta)]$, characteristically encountered in the hydrodynamic theory of lubrication. The equation of convective diffusion contains one more space coordinate than previously;

$$\varepsilon^2\left[\frac{\partial c}{\partial t}+\frac{u_\theta^*}{(1+\varepsilon\zeta)}\frac{\partial c}{\partial\theta}+\frac{u_\varphi^*}{\sin\theta\,(1+\varepsilon\zeta)}\frac{\partial c}{\partial\varphi}\right]+\varepsilon\,\mathrm{Pe}\left[\frac{u_\theta}{(1+\varepsilon\zeta)}\frac{\partial c}{\partial\theta}+\right.$$
$$\left.+\frac{u_\varphi}{\sin\theta\,(1+\varepsilon\zeta)}\frac{\partial c}{\partial\varphi}\right]=\frac{\partial^2 c}{\partial\zeta^2}+\frac{2\varepsilon}{(1+\varepsilon\zeta)}\frac{\partial c}{\partial\zeta}+$$
$$+\frac{\varepsilon^2}{\sin\theta\,(1+\varepsilon\zeta)^2}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial c}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2 c}{\partial\varphi^2}\right] \tag{3.1}$$
$$\left(\mathrm{Pe}=UR\varepsilon/D,\ t=D\tau/R^2,\ u_\varphi^*=\langle U_\varphi\rangle R/D,\ u_\theta^*=\langle U_\theta\rangle R/D\right).$$

Thus, its reduction to the dispersion theory approximation allows us to find an equation which contains two space coordinates and three dispersion factors. The first two correspond to the effective diffusion over the axes φ and θ, while the third describes the crossover effect. The calculations are somewhat more complicated than in part 1, but the method is the same. As a result, taking into account the continuity equation, the condition corresponding to impermeability of the surface of the sphere or an impurity, and the analogous condition (the analog of the second equation of (1.3)) for the external boundary of the region, we obtain

$$\frac{\partial G}{\partial t}+u_\theta^*\frac{\partial G}{\partial\theta}+\frac{u_\varphi^*}{\sin\theta}\frac{\partial G}{\partial\varphi}=\frac{1}{h(\varphi,\theta)}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[h(\varphi,\theta)\sin\theta\,(\mathrm{Pe}^2D_{\theta\theta}(\varphi,\theta)+\right.\right.$$
$$\left.+1)\frac{\partial G}{\partial\theta}\right]+\frac{1}{\sin^2\theta}\frac{\partial}{\partial\varphi}\left[h(\varphi,\theta)(\mathrm{Pe}^2D_{\varphi\varphi}(\varphi,\theta)+1)\frac{\partial G}{\partial\varphi}\right]+$$
$$\left.+\frac{\mathrm{Pe}^2}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(h(\varphi,\theta)D_{\theta\varphi}(\varphi,\theta)\frac{\partial G}{\partial\varphi}\right)+\frac{\partial}{\partial\varphi}\left(h(\varphi,\theta)D_{\varphi\theta}(\varphi,\theta)\frac{\partial G}{\partial\theta}\right)\right]\right\}, \tag{3.2}$$

where

$$D_{ij}(\varphi,\theta)=\frac{1}{h}\int_0^h\Phi_i\Phi_j d\zeta,\quad \Phi_i=\int_0^\zeta u_i(\varphi,\theta,\zeta)\,d\zeta,\quad i,j=\varphi,\theta, \tag{3.3}$$

i.e., the subscripts i and j in (3.3) can take any value of φ and/or θ. Also, $D_{\theta\varphi} = D_{\varphi\theta}$. It should be noted that the crossover phenomenon is related only to the convective motion of the fluid (not the molecular analog in (3.2)).

In the general case, the analytical analysis of Eq. (3.2) is fairly complex. Possible simplifications of (3.2) will be discussed in part 4 when examples are presented. For now, we note that, as in part 2, it is easy to determine the equilibrium concentration at $t \to \infty$. This is found from a formula similar to (2.4):

$$G_\infty=\int_0^{2\pi}d\varphi\int_0^\pi G_0(\varphi,\theta)\,h(\varphi,\theta)\sin\theta\,d\theta\Big/\int_0^{2\pi}d\varphi\int_0^\pi h(\varphi,\theta)\sin\theta\,d\theta.$$

Here, G_0 is the value of G at $t = 0$. The latter is found from (1.8) with natural relabeling of the variables.

4. Examples. The above-obtained basic dispersion equations (1.6), (2.2), and (3.2) contain coefficients of effective diffusion. These coefficients can be determined if we know the solution of the hydrodynamic problem. We will henceforth deal only with the convective part of the dispersion factors, since the remaining part is easily found for a region of known

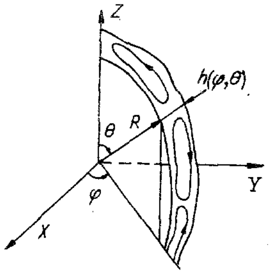

Fig. 3

geometry (a known function h), i.e., we will be concerned with coefficients which contain the multiplier Pe^2 in the corresponding equations. The values of the dispersion factors will be presented in dimensional form so as to more accurately reflect the effect of given parameters on the multiplier.

As the first example, we will examine the well-known problem studied by Reynolds and concerning the motion of a layer of lubricant between flat plates. This corresponds to the problem in part 1 with a linear function h(x). Here, we will use the Reynolds solution [6] written in dimensional variables. Let the XOZ plane be shifted in the direction of the X axis at the rate U. Located a certain distance from this plane is a plate of infinite width and finite length. The plate is inclined at an angle α to the XOZ plane. The flow region between the plate and the plane and the corresponding parameters are shown in Fig. 4, here h_1 and h_2 are the thicknesses of the layer at the left and right ends of the plate. Thus, we find the layer thickness h that

$$h = h_1 - X\,\mathrm{tg}\,\alpha = h_1 - mX,\ m = \mathrm{tg}\,\alpha = (h_1 - h_2)/l.$$

The characteristics of the flow are naturally independent of the Z coordinate.

The solution of the hydrodynamic problem in the lubrication theory approximation is well-known [6]. We will present the characteristics of this solution that are needed for further analysis. The velocity profile over the X axis

$$U_x = U(1 - Y/h) - (2\mu)^{-1}(dp/dX)(Yh - Y^2), \tag{4.1}$$

which gives us the following expressions for mean velocity

$$\langle U_x \rangle = \frac{1}{h}\int_0^h U_x dY = \frac{U}{2} - \frac{h^2}{12\mu}\frac{dp}{dX} \tag{4.2}$$

and the flow velocities relative to the mean value

$$u_x = U_x - \langle U_x \rangle = U(1/2 - \nu) + h^2(dp/dX)(1 - 6\nu + 6\nu^2)/12\mu \tag{4.3}$$

($\nu = Y/h$, while μ is the absolute viscosity of the fluid). The gradient of pressure p is determined by the formula [6]

$$\frac{dp}{dX} = \frac{6\mu U}{h^2}\left(1 - \frac{h^*}{h}\right),\quad h^* = \frac{2h_1h_2}{h_1 + h_2}. \tag{4.4}$$

For the function Φ_x, determined by (1.5), we have

$$\begin{gathered}\Phi_x(\nu) = A_x\nu(1 - \nu) - B_x\nu(1 - \nu)(2\nu - 1),\\ A_x = Uh/2,\ B_x = (h^3/12\mu)dp/dX.\end{gathered} \tag{4.5}$$

Calculation of the convective part of the dispersion factor from Eq. (1.7) with allowance for (4.5) leads to the expression

$$D_x(X) = \frac{1}{30D}\left(A_x^2 + \frac{1}{7}B_x^2\right) = \frac{U^2h^2}{120D}\left[1 + \frac{1}{7}\left(1 - \frac{h^*}{h}\right)^2\right]. \tag{4.6}$$

It should be noted that although Eqs. (4.1-4.4) constitute the zeroth approximation (with respect to ε) of the solution of the hydrodynamic problem, they are quite adequate for

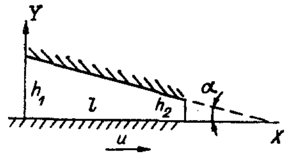

Fig. 4

calculating the dispersion factor in this approximation. The same applies to the other examples illustrating the given point.

It is easily seen that an increase in X is accompanied by a monotonic decrease in D_X. Meanwhile, the ratio of these coefficients at the end points of the interval $X \in (0, \ell)$ is equal to $(h_1/h_2)^2$. In the special case of constant h ($h = h_1 = h_2 = \text{const}$), the coefficient D_X is also constant.

As the second example, we will examine the dispersion of a substance in the gap of a cylindrical bearing. The hydrodynamic part of the problem was studied by Sommerfeld and was described in detail in [7]. If the external surface of the flow region is represented as a circle positioned eccentrically relative to the internal surface, then to within terms of the second order with respect to ε the equation of the external surface will be [7] (Fig. 5)

$$h(\varphi) = \varepsilon(1 - \lambda \cos \varphi),\ \varphi \in (0, 2\pi), \tag{4.7}$$

where $\lambda = e/\varepsilon$; $\varepsilon = R' - R$, i.e., ε is a dimensional parameter differing from the parameter in part 2 by the multiplier R. It is assumed that the length of the cylinders in the axial directions makes it possible to ignore and effects. Meanwhile, the internal cylinder is rotated clockwise with the angular velocity ω.

As was noted in [7], the relations which take the place of (4.1-4.4) – and, thus, (4.5-4.6) – keep their form in the given example with the substitutions

$$U \to \omega R,\ \frac{dp}{dX} \to \frac{1}{R}\frac{dp}{d\varphi},\ h^* \to \frac{2Q}{\omega R} = \frac{2\varepsilon(1-\lambda^2)}{2+\lambda^2}$$

(Q is the rate of fluid flow through the gap), as well as the replacement of the subscript x by ϕ in the quantities A_X, B_X, Φ_X, and D_X. Thus, the value of the dispersion factor is obtained frlom Eq. (4.6) with the given substitutions.

The third example corresponds to the problem in part 3 and pertains to the dispersion of a substance in a gap of a spherical bearing, between eccentrically positioned spheres. The internal sphere rotates, while the external sphere is stationary. The hydrodynamic problem was solved in a lubrication theory approximation in [7]. More precisely, the author of [7] solved a somewhat more general problem in which the internal sphere underwent translational motion. This circumstance leads to a change in the geometry of the flow region, i.e., whereas the instantaneous velocity field is used instead of calculation to determine the resultant and the moment of the force acting on the internal sphere, a change in the boundaries of the region for problems involving heat- and mass transfer occurring within the gap in a certain time intercval necessarily complicates the analysis (this complication being connected with allowing for a new time scale for the change in the form of the region). However, we will not consider the change in the geometry of the region in the present study. Instead, a simplification will be made in the corresponding formulas when the results from [7] are used below. Specifically, we will assume that the vector of translational velocity is equal to zero.

Following [7], we will examine the motion of a viscous incompressible fluid in the gap between two eccentrically positioned spheres with centers O, O' (Fig. 6) and radii R and R' ($R' > R$). The difference in the radii $\varepsilon = R' - R$ will be assumed to satisfy the inequalities ε/R, $\varepsilon/R' \ll 1$. We will use a spherical coordinate R, θ, φ, having drawn the OZ axis through the centers of the spheres as the polar axis. Meanwhile, the OXZ plane of the reference point for longitude φ is drawn through the vector of angular velocity ω (in those cases when the vector ω is parallel to the axis Z, the reading of φ is arbitrary) (Fig. 6). The transverse dimension of the cavity at a certain point M, determined by the coordinates θ and φ, will be designated as h(φ, θ) and be found as the distance between points M and M' on internal and external spheres located on the same radius drawn from the center O of the internal sphere. Then, with an error of the order of $(\varepsilon/R)^2$, we obtain an equality which is analogous to (4.7):

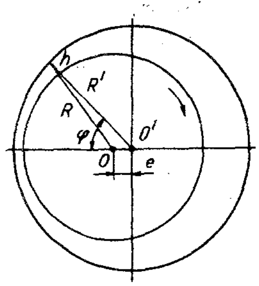

Fig. 5

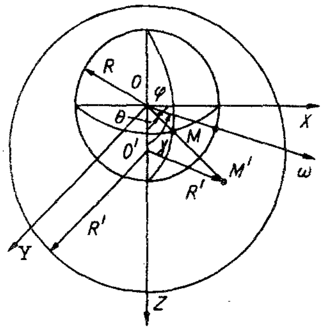

Fig. 6

$h = \varepsilon(1 + \lambda\cos\theta)$, $\theta \in (0, \pi)$, i.e., h is independent of φ. In this case, eccentricity e is determined as the distance between the centers of the spheres OO' (Fig. 6).

In accordance with the results obtained in [7], the structure of the solution of the hydrodynamic problem is similar to Eqs. (4.1-4.3). The main difference is that there are now two velocity components U_φ and U_θ, two components of mean velocity, etc. Thus, for the quantities corresponding to the coordinates φ and θ, we need to make the following substitutions in Eqs. (4.1-4.3)

$$u_x \text{ for } \begin{matrix} u_\varphi \\ u_\theta \end{matrix}, \quad \frac{dp}{dX} \text{ for } \begin{matrix} (R\sin\theta)^{-1}\partial p/\partial\varphi \\ R^{-1}\partial p/\partial\theta \end{matrix}, \quad U \text{ for } \begin{matrix} U_\varphi^0 \\ U_\theta^0 \end{matrix}, \quad A_x \text{ for } \begin{matrix} A_\varphi \\ A_\theta \end{matrix}, \quad B_x \text{ for } \begin{matrix} B_\varphi \\ B_\theta \end{matrix},$$
$$\Phi_x \text{ for } \begin{matrix} \Phi_\varphi \\ \Phi_\theta \end{matrix}.$$

With allowance for the above simplifications, the corresponding formulas [7] for pressure p and the velocity components U_φ^0, U_θ^0 reduce to the form

$$p = -\frac{6\mu R^2\omega e\sin\gamma\,(2+\lambda\cos\theta)\sin\theta\sin\varphi}{\varepsilon^3(4+\lambda^2)(1+\lambda\cos\theta)^2}, \tag{4.8}$$
$$U_\varphi^0 = \omega R\,[\cos\gamma\sin\theta - \sin\gamma\cos\theta\cos\varphi], \quad U_\theta^0 = -\,\omega R\sin\gamma\sin\varphi$$

(ω is the magnitude of the vector of angular velocity; γ is the angle formed by the polar axis and the vector ω).

Having calculated the derivatives of pressure p (4.8) and having inserted them into Eqs. (3.3), we obtain Eqs. (4.6) with the following changes in notation for the diagonal elements of the dispersion-factor matrix:

$$D_{\varphi\varphi} = \frac{[\omega R\varepsilon(1+\lambda\cos\theta)]^2}{120D}\left\{(\cos\gamma\sin\theta - \sin\gamma\cos\theta\cos\varphi)^2 + \right.$$
$$\left. + \frac{[\lambda\sin\gamma\cos\varphi\,(2+\lambda\cos\theta)]^2}{7(4+\lambda^2)^2}\right\}, \tag{4.9}$$
$$D_{\theta\theta} = \frac{[\omega R\varepsilon\sin\gamma\sin\varphi]^2}{120D}\left[\frac{\lambda^2(3\lambda+2\cos\theta+\lambda^2\cos\theta)^2}{7(4+\lambda^2)^2} + (1+\lambda\cos\theta)^2\right],$$

while for the crossover matrix $D_{\theta\varphi}$ we obtain the formula

$$D_{\theta\varphi} = \frac{1}{30D}\left(A_\theta A_\varphi + \frac{B_\theta B_\varphi}{7}\right) = \frac{[\omega R\varepsilon\sin\gamma\,(1+\lambda\cos\theta)]^2\sin\varphi}{120D}\times \tag{4.10}$$
$$\times\left[\frac{\lambda^2\cos\varphi\,(2+\lambda\cos\theta)(3\lambda+2\cos\theta+\lambda^2\cos\theta)}{7(1+\lambda\cos\theta)(4+\lambda^2)^2} + \cos\theta\cos\varphi - \operatorname{ctg}\gamma\sin\theta\right].$$

Equations (4.9-4.10) are simplified considerably when the polar axis coincides with the direction of the axis of rotation. In this case, sin γ = 0 and we find from (4.9-4.10) that

$D_{\theta\varphi} = 0$, $D_{\theta\theta} = 0$, $D_{\varphi\varphi} = [\omega R\varepsilon\ (1 + \lambda\cos\ \theta)\ \sin\ \theta]^2/120D$. Then it is easily seen that $u^*_\theta = 0$, so that (3.2) is in essence reduced to unidimensional equation (2.5) with the corresponding transformations of variables. This occurs because we now consider only the dispersion terms in (3.2) that contain the multiplier Pe^2. Here, the variable θ begins to act as a parameter in the equation. Another possible simplification of (3.2) is the existence of concentric spheres, i.e., $\lambda = 0$. In this case, the equation reduces to the case already examined, since the axis of rotation becomes the only direction in the problem. As a result, it is best to take this axis as the polar axis, i.e., we take $\gamma = 0$. It is assumed that the vector of angular velocity does not change over time in this instance. Otherwise, ω and γ and the reference point for longitude φ would be functions of time. In the latter case, φ would be replaced by $\varphi - \varphi_0(t)$ in all of the formulas. This situation would require substantiation of both the hydrodynamic relations for the velocity components, since they were found from the solution of the steady-state problem, and the relations for the scales of the characteristic times in the mass transfer problem. It is clear that these equations can exist if the vector ω changes slowly enough.

5. Remarks. The example discussed in part 4 dealt with partial differential equations with variable coefficients, which are fairly complex for the purposes of exact analysis. However, in the case of a small degree of eccentricity etc. — more exactly, at $\lambda \ll 1$ — perturbation methods can be effectively used to find an approximate solution. It should be noted that the dispersion factors D_X, D_φ, etc. have no singularities within the domains. In this case, the zeroth-approximation equations for the examples in part 4 essentially reduce to (2.5), while the subsequent approximations will contain source terms (which do not especially complicate the construction of the solutions). In the case of periodic boundary conditions (2.6), Eq. (2.7) serves as the reference solution for the zeroth approximation.

N. E. Zhukov and S. A. Chaplygin used a biharmonic equation for the stream function ψ to obtain the solution of the hydrodynamic problem without simplifying assumptions from lubrication theory [6, 15] for a cylindrical bearing during the flow of a lubricant layer at low Reynolds numbers. They used bipolar coordinates ξ and η, which are connected with the Cartesian coordinates by the relations $X = a\ \sinh\ \xi/(\cosh\ \eta - \cos\ \xi)$, $Y = a\ \sin\ \xi/(\cosh\ \eta - \cos\ \xi)$ (a is the geometric parameter). Here, the internal and external cylinders correspond to the coordinates η_0 and η_1, respectively. The dispersion relation for analyzing mass transfer in the given cylindrical gap can be found by the method substantiated above or by the method used in [10-12]. This relation has the form

$$b(\xi)\frac{\partial G}{\partial t} + Q\frac{\partial G}{\partial \xi} = \frac{\partial}{\partial \xi}\left[(\mathrm{Pe}^2 D_\xi(\xi) + 1)\frac{\partial G}{\partial \xi}\right], \quad (5.1)$$

where

$$D_\xi(\xi) = \int_{\eta_0}^{\eta_1} \psi_*^2 d\eta, \quad b(\xi) = \int_{\eta_0}^{\eta_1} \frac{d\eta}{(\mathrm{ch}\ \eta - \cos\ \xi)^2}, \quad (5.2)$$

while ψ_* is the stream function minus its component for average motion. The relation is easily found from the formulas presented in [6, 15], these equations being omitted here due to their cumbersome nature. This last fact complicates calculation of the first integral of (5.2) (the second is considered to be elementary). This integral is probably best found by a numerical method in the course of solving heat- and mass transfer problems on the basis of Eq. (5.1).

As regards the applicability of the proposed equations (as in dispersion theory), it must be noted that dispersion theory equations have been fully validated only for the simplest cases involving flow in circular and "planar" prismatic tubes [16, 17]. In these studies, the authors first obtained the exact solution and then derived the dispersion equation by taking the limit. In the overwhelming majority of studies in dispersion theory, different procedures are used for expansions into series etc., without a mathematically clear indication of the range of applicability of the expansions. Thus, at present the evidence for the validity of dispersion theory is more physical than mathematical in nature.

We believe that the construction of dispersion equations on the basis of the small-parameter method has certain advantages compared to other approaches and is applicable within a broader range of parameters and variables. This method has proven itself in several problems of dispersion theory and has generally produced the same results as the other methods. In addition, the small-parameter method is a natural yet systematic approach to the construction of corrections for the main approximation in accordance with established perturbation techniques. All this suggests that the small-parameter method can be used in problems of the given type and that the resulting equations can be used for cases of moderate or large values of ε. This is even more the case, in light of the fact that the quantity which is of physical importance is the product of the corresponding dimensionless derivative and a certain power of the above parameter. The asymptotic character of the dispersion-based description of processes – established in the literature [1, 2, etc.] – means that the corresponding derivatives in the equations which are analogous to (1.1), (2.1), and (3.1) decrease over time.

The features just discussed are closely allied with the relationship between the mean and characteristic transfer rates. In accordance with (1.1) etc., the ratio of these rates should be of the order of ε for the conclusions of the theory to be valid. However, it should be noted that only terms in (1.1) with the multipliers Pe participate in the formation of dispersion effects. The other, sign-changing terms are to a certain degree averaged (see (1.5), (1.7), for example) and accompany the second derivative with respect to the coordinates in final dispersion equation (1.6). To a certain extent, this makes the transport of material by the flow at the mean velocity equal in importance to transport by the dispersion mechanism.

Mention should also be made of new results obtained regarding the construction of dispersion factors for flow between coaxial cylinders along the cylinder axis as well as in the direction φ [18]. Three independent components of the matrix of effective diffusion coefficients were obtained in the latter study.

LITERATURE CITED

1. G. Taylor, "Dispersion of soluble matter in solvent flowing through a tube," Proc. R. Soc. London Ser. A, 219, No. 1137 (1953).
2. V. V. Dil'man and A. E. Kronberg, "Relaxational phenomena during longitudinal mixing" Teor. Osn. Khim. Tekhnol., 17, No. 5 (1983).
3. M. E. Erdogan and P. C. Chatwin, "The effects of curvature and buoyancy on the laminar dispersion of solute in a horizontal tube," J. Fluid Mech., 29, Part 3 (1967).
4. M. Jonson and R. D. Kamm, "Numerical studies of steady flow dispersion. A low Dean number in a gently curving tube," J. Fluid Mech., 172 (1986).
5. Ph. Daskopoulos and A. M. Lenhoff, "Dispersion coefficient for laminar flow in curved tubes," AIChE J., 34, No. 12 (1988).
6. N. A. Slezkin, Dynamics of Incompressible Viscous Media [in Russian], GITTL, Moscow (1965).
7. L. G. Loitsyanskii, Mechanics of Liquids and Gases, Nauka, Moscow (1987).
8. A. Naife, Introduction to Perturbation Methods [Russian translation], Mir, Moscow (1984).
9. J. Cowell, Perturbation Methods in Applied Mathematics [Russian translation], Mir, Moscow (1972).
10. A. I. Moshinskii, "Effective diffusion of a dynamically passive impurity in narrow channels," Prikl. Mekh. Tekh. Fiz., No. 3 (1987).
11. A. I. Moshinskii, "Certain aspects of the dispersion of an impurity in a circular tube," Inzh. Fiz. Zh., 53, No. 2 (1987).
12. A. I. Moshinskii, "Diffusion model in the presence of circulation zones in process equipment," Teor. Osn. Khim. Tekhnol., 22, No. 3 (1988).
13. V. A. Yakubovich and V. M. Starzhinskii, Linear Differential Equations with Periodic Coefficients and Their Application [in Russian], Nauka, Moscow (1972).
14. H. Brenner, "The diffusion model of longitudinal mixing in beds of finite length. Numerical values," Chem. Eng. Sci., 17, 229 (1962).
15. N. E. Zhukovskii and S. A. Chaplygin, "Friction of a lubricant layer between a journal and a bearing," in: Collected Works on Mechanics and Mathematics [in Russian], GITTL, Moscow (1954).
16. Ya. S. Ufland, "Exact solution of a problem of nonsteady convective diffusion in a cylinder," Zh. Tekh. Fiz., 57, No. 2 (1987).
17. Ya. S. Ufland, "Exact solution of the problem of nonsteady convective heat transfer in a plane channel," Inzh. Fiz. Zh., 54, No. 6 (1988).
18. R. Smith, "Shear along a rotating axle in a closely fitting shaft," J. Fluid Mech., 219, 647 (1990).